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FREE SETS FOR NOWHERE-DENSE SET MAPPINGS

BY
URI AVRAHAM

ABSTRACT
We give some consistency results on the existence of uncountable free sets for nowhere-dense set mappings. For example, we prove that it is relatively consistent with ZFC and Martin's Axiom that any nowhere-dense set mapping defined on the reals has an uncountable free set.

§0. Let A be a set, a function $f: A \to P(A)$ is called a *set mapping*. ($P(A)$ is the power set of A.) $X \subseteq A$ is said to be *free* iff for all $x, y \in X$, $x \neq y \Rightarrow x \notin f(y)$. A classical result says that if κ is an infinite cardinal $< |A|$ and $|f(a)| < \kappa$ for all $a \in A$, then a free set of cardinality $|A|$ exists (Lázár and Hajnal, see [5]). The game is to require $f(a)$ to be small and to ask for a large free set. We will concentrate in the case where f is defined on the set of real numbers R, and $f(r)$ is nowhere-dense for all $r \in R$ (call such f a *nowhere-dense set mapping*) and we ask for an uncountable free set. This type of question, requiring $f(a)$ to be small in another sense than cardinality, is very interesting because it leads to consistency results. It was asked by P. Erdös and A. Hajnal in their list of unsolved problems [2]; problem 38A reads: Assume that f is nowhere dense in R. Does there then exist a free subset of power $\aleph_1$? S. Hechler gave a consistency answer, assuming CH (continuum hypothesis) he showed there exists a nowhere-dense set mapping f defined on R with no uncountable free set ($f(r)$ is even a monotonic sequence converging to r). See [3]. On the other hand, [4] remarks the following. If one adds $\aleph_2$ Cohen reals (forcing with finite conditions) then the set G of the Cohen reals added is a Luzin set: For any set F of first category $F \cap G$ is countable. So, by restricting to the set G, we get that in this model every set mapping f such that $f(r)$ is of first category for all $r \in R$, has a free set which is of second category and of power the continuum. This follows by Lázár–Hajnal because, restricted to G, the notion of "first category" and the notion of "countable cardinality" coincide. But this is also

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why that answer is not completely satisfactory — the whole motivation of the question was to ask something different from the question answered by Lázár and Hajnal. So, in §2 we show the consistency of ZFC with Martin's Axiom and $2^{\aleph_0} = \aleph_2$ (hence there are no Luzin sets) of

(*) *Every nowhere-dense set mapping has an uncountable free set.*

In §1 we show the consistency of ZFC with non CH ($2^{\aleph_0}$ having any reasonable value) with the negation of (*). This was asked by Hechler in [3] and [4].

As an appetizer let me say that any question you can ask seems to be untouched: Get a model with no Luzin sets and any nowhere-dense set mapping has a free set of cardinality continuum. Ask the free sets to be of second category (always with no Lusin sets). What with set mapping having as values first category set? Measure zero sets? Get the consistency of Martin's Axiom with the negation of (*). What about functions defined on pairs? etc.

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§1. A nowhere-dense set mapping with no uncountable free set is consistent with $\neg$CH

We assume in this section that $\neg$CH holds and our aim is to add (generically) a nowhere-dense set mapping f which does not have uncountable free set. As our forcing posets add new reals, it is necessary to iterate the generic extensions so that finally f will be defined on all the reals. Let us first describe how the nowhere-dense set $f(r)$ is generically obtained for a particular real r.

For $r \in R$, P_r is the natural forcing poset for adding a nowhere-dense set which includes r, so a condition in P_r gives a finite number of reals which are to be in the nowhere-dense set, and a finite number of open (rational) intervals included in the complement of that set. More precisely:

1.1. Definition. P_r is the collection of all $p = (p', p^*)$ such that $p' \subseteq R$ is finite with $r \in p'$, and p^* is a finite collection of (nonempty) rational intervals such that $p' \cap (\bigcup p^*) = \emptyset$ (in particular $r \notin p^*$, this is the main point). The partial order is defined naturally by $p = (p', p^*) \leqq q = (q', q^*)$ (q is more informative than p) iff $p' \subseteq q$ and $\bigcup p^* \subset \bigcup q^*$.

Let $\dot{P}_r$ be a V-generic filter over P_r. It is readily checked that for each interval I, $\{p = (p', p^*) \in P_r \mid \text{for some rational interval } J \subseteq I, J \in p^*\}$ is a dense subset of P_r. Hence $\bigcup \{p' \mid (p', p^*) \in \dot{P}_r \text{ for some } p^*\}$ is a nowhere-dense set, denote this set by $f(r)$ and you can see how our generic set mapping function f with no free uncountable sets is constructed. For any $p, q \in P_r$, if $p^* = q^*$ then p and q are compatible by $(p' \cup q', p^*)$. Hence, as there are only countably many rational intervals, P_r satisfies the c.a.c. (countable antichain condition — any collection of pairwise incompatible members of P_r is countable). Moreover P_r satisfies property K; any collection of cardinality $\aleph_1$ of members of P_r contains an uncountable subcollection every two members of which are compatible. So forcing with P_r does not collapse cardinals. It is also clear that forcing with P_r adds a new real. (The set of rational intervals which belong to p^* for some $p \in \dot{P}_r$ is not in the ground model.)

1.2. Definition. Let $S \subseteq R$ be a set of reals. P_S is the collection of all finite functions h defined on S such that for $r \in$ domain of h, $h(r) \in P_r$. The partial order is defined componentwise, i.e., $h \leqq g$ iff $\mathrm{Dom}(h) \subseteq \mathrm{Dom}(g)$ and for $r \in \mathrm{Dom}(h)$, $h(r) \leqq g(r)$ in P_r.

The proof that P_S satisfies property K uses the classical Δ-system argument applied to the domain of the functions involved (see [5] appendix). We also need the fact that iteration — taking direct limits at limit stages — of posets satisfying property K, satisfies property K (see [6]).

Now if $\dot{P}_S$ is V-generic filter over P_S, we get for $r \in S$ a generic filter over P_r, namely $\dot{P}_r = \{h(r) \mid h \in \dot{P}_S\}$ which we call $\dot{P}_S \cap P_r$. The nowhere-dense set thus obtained from $\dot{P}_r$ is denoted by $f(r)$, so the domain of f is S.

To make the proof easier, let us first prove that in V^{P_S} the function f does not have an uncountable free set, then we will have to iterate these forcings so as to define f on all of the reals. So suppose $p \Vdash$ "$\langle r_i \mid i < \omega_1 \rangle$ is a free set for f". For each $i < \omega_1$ pick $p_i \geqq p$ and a real $s_i \in S$ such that $p_i \Vdash$ "$r_i = s_i$", by enlarging p_i further we can assume that $s_i \in \mathrm{Dom}(p_i)$. Now find an uncountable $J \subseteq \omega_1$ and a finite sequence $\langle \bar{I}_1, \cdots, \bar{I}_n \rangle$ of sets of rational intervals such that

(1) $\{\mathrm{Dom}(p_i) \mid i \in J\}$ is a Δ-system, say $\mathrm{Dom}(p_i) = \{a^i_1, \cdots, a^i_n\}$ and $s_i = a^i_1$, and for $i, j \in J$, $a^i_l = a^j_m \Rightarrow l = m$.

(2) $p_i(a^i_k) = (p'_{i,k}, \bar{I}_k)$. (By definition we also have $a^i_k \in p'_{i,k}$.)

Now take any $i \neq j$ in J.

Define $p_i \cup p_j$ as follows. $D = \mathrm{Dom}(p_i \cup p_j) = \mathrm{Dom}(p_i) \cup \mathrm{Dom}(p_j)$ and for $a \in D$ if $a \in \mathrm{Dom}(p_i) \cap \mathrm{Dom}(p_j)$ then let $p_i(a) = (X, \bar{I}_k)$ it follows that $p_j(a) = (Y, \bar{I}_k)$, then we have $(p_i \cup p_j)(a) = (X \cup Y, \bar{I}_k)$. In case $a \in \mathrm{Dom}(p_i) - \mathrm{Dom}(p_j)(\mathrm{Dom}(p_j) - \mathrm{Dom}(p_i))$ we let $(p_i \cup p_j)(a) = p_i(a)$ $(= p_j(a))$. So $p_i \cup p_j \in P_S$.

Now we define $p^* > p_i \cup p_j$ such that $p^* \Vdash$ "$r_i \in f(r_j)$" and this is a contradiction. For all $a \neq s_j$, $p^*(a) = (p_i \cup p_j)(a)$. But for $a = s_j$, say $(p_i \cup p_j)(s_j) = (Z, \bar{I}_1)$, define $p^*(s_j) = (Z \cup \{s_i\}, \bar{I}_1)$. This is O.K. because $s_i \notin \bigcup \bar{I}_1$ (as $s_i = a^i_1$).

Assume V — the universe we start with — satisfies the negation of CH. We will iterate posets of the kind P_S, starting with P_R, and define the function f on the newly created reals so that finally (after ω_1 many steps) the function f is defined on all the reals and has no uncountable free set (as we shall see).

1.3. Definition of the Iteration. Set $Q_0 = \emptyset$, $S_0 = R$. Define by induction on $\alpha \leqq \omega_1$, $\alpha > 0$, posets Q_α and names S_α in V^{Q_α} such that Q_α is a set of finite functions q defined on finite subsets of α, such that $q(\xi)$ is a name in Q_ξ (for $\xi > 0$), defined by:

(1) $q(0) \in P_R$ (R is the set of reals in V). So Q_1 is essentially P_R.

(2) If α is limit then $q \in Q_\alpha$ iff q is a finite function such that all $\gamma < \alpha$, $q \mid \gamma \in Q_\gamma$. ($q \mid \gamma$ is the restriction of q to γ.)

(3) If $\alpha = \beta + 1$ then Q_α is the set of all functions q such that $\mathrm{Dom}(q) \subseteq \alpha$, $q \mid \beta \in Q_\beta$ and if $\beta \in \mathrm{Dom}(q)$ then $q \mid \beta \Vdash q(\beta) \in P_{S_\beta}$.

(4) S_α is a name in V^{Q_α} such that $\emptyset \Vdash$ "S_α is the set of all reals which are not in V^{Q_β} for any $\beta < \alpha$." In case $\alpha = 1$ we get $\emptyset \Vdash^{Q_1}$ "S_1 is the set of all reals which are not in V".

Finally $Q = Q_{\omega_1}$ is the desired poset. A projection from Q on Q_γ is naturally defined by $q \to q \mid \gamma$.

1.4. Claim. *Any real in V^Q is in V^{Q_γ} for some $\gamma < \omega_1$.*

The proof of the claim is standard and uses the fact that Q satisfies the c.a.c. □ So for any real r in V^Q, looking at the minimal α such that $r \in V^{Q_\alpha}$ we get that either $r \in R^V$ or $r \in S_\alpha$ for some $\alpha < \omega_1$. Hence $f(r)$ is defined at stage $Q_{\alpha+1}$ (or at Q_1 if $r \in R$). $f(r)$ is a nowhere-dense set in V^Q because the property of being a nowhere-dense set is absolute. Before proving that f has no uncountable free set, let us show that the conditions in Q can be assumed to be of a more explicit nature.

1.5. Definition. For each $1 \leqq \alpha < \omega_1$ choose a fixed name $\tau_\alpha \in V^{Q_\alpha}$ such that $\emptyset \Vdash$ "$\tau_\alpha : | S_\alpha | \to S_\alpha$ is a one-to-one onto function". Let $\boldsymbol{h}$ be a name in V^{Q_α}, $q \in Q_\alpha$ such that $q \Vdash$ "$\boldsymbol{h} \in P_{S_\alpha}$", $(q, \boldsymbol{h})$ is *explicit* iff there exists a finite set $\{\xi_1, \cdots, \xi_m\}$ of ordinals and there exists a collection $\{\bar{I}_1, \cdots, \bar{I}_m\}$ such that $\bar{I}_i$ is a finite set of rational intervals, such that

(1) $q \Vdash$ "$\mathrm{Dom}(\boldsymbol{h}) = \{\tau_\alpha(\xi_1), \cdots, \tau_\alpha(\xi_m)\}$" and

(2) $q \Vdash$ "if $\boldsymbol{h}(\tau_\alpha(\xi_i)) = (p', p^*)$ then $p^* = \bar{I}_i$".

1.6. Lemma. *The set of conditions $q \in Q$ such that, for $\alpha \in \mathrm{Dom}(q)$, $(q \mid \alpha, q(\alpha))$ is explicit is a dense set in Q. (We call such q explicit.)*

PROOF. Let $q \in Q$ be given. Inductively we will construct $q = q_0 \leqq q_1 \leqq \cdots \leqq q_n$ a sequence (which will turn out to be finite) increasing in Q, as follows. Assume q_i is constructed. Let $\alpha_i \in \mathrm{Dom}(q_i)$ be the least such that $(q_i \mid \alpha_i, q_i(\alpha_i))$ is not explicit. Now define q_{i+1} such that

(1) $q_{i+1}(\alpha) = q_i(\alpha)$ for all $\alpha > \alpha_i$.

(2) $q_{i+1} > q_i$ and $(q_{i+1} \mid \alpha_i, q_{i+1}(\alpha_i))$ is explicit.

A moment of reflection shows this is possible. As the sequence α_i is decreasing, at some finite i the definition breaks, hence $q_i(\alpha)$ is explicit for all $\alpha \in \mathrm{Dom}(q_i)$ and q_i is as required. □

1.7. MAIN LEMMA. *There is no uncountable free set for f.*

PROOF. Assume to the contrary that $q \Vdash$ "$\langle r_i \mid i < \omega_1 \rangle$ is a free set for f". For each $i < \omega_1$ pick $q'_i \geqq q$ and $\alpha_i < \omega_1$ such that $q'_i \Vdash$ "$r_i \in S_{\alpha_i}$". Then pick $q_i \geqq q'_i$ such that

(1) $\alpha_i \in \mathrm{Dom}(q_i)$ and q_i is explicit.

(2) $q_i \Vdash$ "$r_i \in \mathrm{Dom}(q_i(\alpha_i))$" and even for some ordinal ξ_i, $q_i \Vdash$ "$r_i = \tau_{\alpha_i}(\xi_i) \in \mathrm{Dom}(q_i(\alpha_i))$".

Now we can pick $J \subseteq \omega_1$ uncountable such that

(1) $\{\mathrm{Dom}(q_i) \mid i \in J\}$ is a Δ system with kernel A. $\mathrm{Dom}(q_i) = \{\alpha(0, i), \cdots, \alpha(n, i)\}$ with $\alpha(l, i) < \alpha(l+1, i)$ and $A = \{\alpha(k, i) \mid k < l_0\}$. And $i < j \Rightarrow \alpha(n, i) < \alpha(l_0, j)$.

(2) For $l \leqq n$, as q_i is explicit, we have a finite set of ordinals and a finite collection of finite sets of rational intervals $\{\xi_1^{l,i}, \cdots, \xi_{m_l}^{l,i}\}$ and $\{\bar{I}_1^l, \cdots, \bar{I}_{m_l}^l\}$ as in Definition 1.5 (notice that we wrote m_l and not $m_{l,i}$, $\bar{I}_1^l$ and not $\bar{I}_1^{l,i}$, since we can assume these do not depend on i by the pigeon hole principle).

(3) For any $l \leqq n \langle \{\xi_1^{l,i}, \cdots, \xi_{m_l}^{l,i}\} : i \in J \rangle$ form a Δ system with kernel $\{\xi_k^{l,i} \mid k \in A^l\}$ for some fixed $A^l \subseteq m_l + 1$.

(4) For some fixed $l \leqq n$ and $k \leqq m_l$, for all $i \in J$ $q_i \Vdash$ "$r_i = \tau_\alpha(\xi)$" where $\alpha = \alpha(l, i)$ and $\xi = \xi_k^{l,i}$ (recall we called α, α_i; and ξ, ξ_i).

Now pick $i < j$ in J, we shall define an extension q^* of q_i and q_j such that $q^* \Vdash$ "$r_i \in f(r_j)$" (a contradiction). Set $\mathrm{Dom}(q^*) = \mathrm{Dom}(q_i) \cup \mathrm{Dom}(q_j)$. The definition of $q^*(\beta)$ is inductive. We also prove inductively that $q^* \mid \beta \geqq q_i \mid \beta$, $q_j \mid \beta$. There are four cases:

Case $\beta \in A$ and $\beta \neq \alpha_j$. Define $q^*(\beta) \in V^{Q_\beta}$ such that $q^* \mid \beta \Vdash$ "$q^*(\beta) = q_i(\beta) \cup q_j(\beta)$". To prove that this is possible we must show that $q^* \mid \beta \Vdash$ "$q_i(\beta)$ and $q_j(\beta)$ are compatible"; this follows because q_i and q_j are part of a Δ system and because $\xi \neq \xi' \Rightarrow \tau_\beta(\xi) \neq \tau_\beta(\xi')$.

Case $\beta \in A$ and $\beta = \alpha_j$ (hence also $\beta = \alpha_i$). To shorten notation, let us agree that if $p = (p', p^*) \in P_r$ and s is a real, then $s \in p$ can be written instead of $s \in p'$.

Now in this case, by (4) $q_j \Vdash$ "$r_j = \tau_\beta(\xi_k^{l,j})$", denote $\xi_k^{l,j}$ by ξ. Define $q^*(\beta) \in V^{Q_\beta}$ such that $q^* \mid \beta \Vdash$ "$q^*(\beta) \geqq q_i(\beta), q_j(\beta)$" and such that $q^* \mid \alpha_j \Vdash$ "$\tau_\beta(\xi_k^{l,i}) \in q^*(\beta)(\tau_\beta(\xi))$". For this to work we should show that $q_i \mid \beta \Vdash$ "$\tau_\beta(\xi_k^{l,i}) \not\in \bigcup \bar{I}_k^l$" but this holds by the definition of P_r. Then, when the definition of q^* is completed, we will get

$$q^* \Vdash \text{“}r_i = \tau_{\alpha_i}(\xi_k^{l,i}) \in q^*(\alpha_j)(\tau_\beta(\xi)) = q^*(\alpha_j)(r_j)\text{”}$$

so that $q^* \Vdash$ "$r_i \in f(r_j)$".

Case $\beta \not\in A$ and $\beta \neq \alpha_j$. If $\beta \in \mathrm{Dom}(q_i)$ set $q^*(\beta) = q_i(\beta)$, while if $\beta \in \mathrm{Dom}(q_j)$ set $q^*(\beta) = q_j(\beta)$. Because $\beta \not\in A$, one and only one of these cases occurs.

Case $\beta \not\in A$ and $\beta = \alpha_j$. In this case, as $i < j$, we have $\alpha_i < \alpha_j$. So that in V^{Q_β}, $\tau_{\alpha_i}(\xi_k^{l,i})$ is a real. Hence we can define $q^*(\beta) \in V^{Q_\beta}$ so that $q^* \mid \beta \Vdash$ "$q^*(\beta) \geqq q_j(\beta)$" and $q^* \mid \beta \Vdash$ "$\tau_{\alpha_i}(\xi_k^{l,i}) \in q^*(\beta)(\tau_\beta(\xi_k^{l,j}))$". So finally we will again have $q^* \Vdash r_i \in f(r_j)$. □

§2. Results with Martin's Axiom

2.1. DEFINITION. We call a set mapping f a *sequence mapping* iff for all $r \in R$, $f(r)$ is a sequence with limit r.

We are interested in free sets for sequence mapping defined on set of reals of cardinality $\aleph_1$, because on higher cardinalities we can use the Lázár–Hajnal theorem mentioned in the introduction.

2.2. THEOREM. *Martin's Axiom $+ 2^{\aleph_0} > \aleph_1$ implies that whenever f is a sequence mapping defined on uncountable $A \subseteq R$, there is a subset of A free for f and uncountable.*

PROOF. Let $P = \{p \subseteq A \mid p$ is finite and free for $f\}$. First we prove that P satisfies the c.a.c. So we are given a collection $\{p_\alpha \mid \alpha < \omega_1\} \subseteq P$ and have to find p_α and p_β compatible. Uniformise the p_α as possible, so we can assume for some $n < \omega$ $|p_\alpha| = n$ for $\alpha < \omega_1$, also assume the p_α form a Δ system, then by throwing out the common part of the Δ system we can assume the p_α are actually disjoint. Set $p_\alpha = \{a_1^\alpha < a_2^\alpha < \cdots < a_n^\alpha\}$ (increasing real numbers) and choose rational intervals I_i^α, $1 \leqq i \leqq n$, such that $a_i^\alpha \in I_i^\alpha$ and $I_i^\alpha \cap f(\alpha_j^\alpha) = \varnothing$. By taking a subcollection and reenumerating it we assume $I_i^\alpha = I_i$ for all $\alpha < \omega_1$, $i = 1, \cdots, n$ (i.e. a fixed sequence of rational intervals is good for all $\alpha < \omega_1$ and hence for $i \neq j$, $a_i^\alpha \notin f(a_j^\beta)$ because $a_i^\alpha \in I_i = I_i^\beta$ is disjoint from $f(a_j^\beta)$).

Now we define by induction on $i \leqq n$ uncountable sets $A_i, B_i \subseteq \omega_1$ such that for $i > 0$

$$\alpha \in A_i \quad \text{and} \quad \beta \in B_i \Rightarrow \{a_1^\alpha, \cdots, a_i^\alpha\} \cup \{a_1^\beta, \cdots, a_i^\beta\} \in P.$$

The definition is as follows: $A_0 = B_0 = \omega_1$. Suppose A_{i-1}, B_{i-1} are constructed, for each $\beta \in B_{i-1}$ find a rational interval J_β such that $J_\beta \cap f(a_i^\beta) = \emptyset$ and $J_\beta \cap \{a_i^\alpha \mid \alpha \in A_{i-1}\}$ is uncountable. (Here we use the fact that f is a sequence mapping and A_{i-1} is uncountable.) Then define uncountable $B'_i \subseteq B_{i-1}$ such that for some J, $J_\beta = J$ for all $\beta \in B'_i$. Next, for any $a_i^\alpha \in J$ $\alpha \in A_{i-1}$ (and there are uncountably many such a_i^α) choose a rational interval I_α with $I_\alpha \cap f(a_i^\alpha) = \emptyset$ and $I_\alpha \cap \{a_i^\beta \mid \beta \in B'_i\}$ uncountable. Now find uncountable $A_i \subseteq A_{i-1}$ such that for some I, $I = I_\alpha$ for all $\alpha \in A_i$. Set $B_i = \{\beta \in B'_i \mid a_i^\beta \in I\}$, then B_i is uncountable. It follows that for $\alpha \in A_i$ and $\beta \in B_i$, $a_i^\alpha \notin f(a_i^\beta)$ and $a_i^\beta \notin f(a_i^\alpha)$. Hence $\{a_1^\alpha, \cdots, a_i^\alpha\} \cup \{a_1^\beta, \cdots, a_i^\beta\} \in P$.

Before applying Martin's Axiom we have to specify what are the dense sets we use, that will provide us with an uncountable filter $G \subseteq P$ and then $\bigcup G$ is uncountable and free. Actually we find $p \in P$ and look at $\{p' \in P \mid p' \geqq p\} = P'$.

The following lemma was used in [1] in a similar context.

2.3. Lemma. *Let P be an uncountable poset satisfying the* c.a.c. *Then there exists $p \in P$ such that, setting $P' = \{p' \in P \mid p' \geqq p\}$, every generic filter over P' is uncountable.*

Proof. It is enough to find $p \in P$ which forces the generic filter to be uncountable. Assume, in order to get a contradiction, that for any $p \in P$, $p \Vdash$ "the generic filter is countable". Then there exists a name f such that for any $p \in P$, $p \Vdash$ "$f : \aleph_0 \to G$ is onto" where G is the canonical name of the generic filter. Now because P satisfies the c.a.c. we get a countable set C such that for any $n < \aleph_0$, if for some $p^*, p \in P$, $p \Vdash$ "$f(n) = p^*$" then $p^* \in C$. Any $p^* \in P - C$ gives a contradiction because $p^* \Vdash$ "$p^* \in G$" hence for some $n < \aleph_0$ and $p \geqq p^*$, $p \Vdash$ "$f(n) = p^*$". □

Now we return our attention to the case f is a nowhere-dense set mapping.

2.4. Theorem. *The following is consistent with* ZFC.

$$2^{\aleph_0} = \aleph_2 + \textit{Martin's Axiom} + (*).$$

PROOF. We start with $V = L$ so that $\diamondsuit_{\omega_2}$ holds. The iteration will be an ω_2 long iteration of c.a.c. posets of cardinality $\aleph_1$ so that CH holds at each stage of the iteration. The diamond is used to tell which is the poset to force with next. All this will be made clearer with more details; let us concentrate now on the successor stage.

2.5. LEMMA. *Assume* CH. *Let f be a nowhere-dense set mapping. Then there exists a* c.a.c. *poset P of cardinality $\aleph_1$ such that in V^P f has an uncountable free set.*

PROOF. We can assume $f(r)$ is closed (simply take the closure of $f(r)$). The only use of CH is to obtain a Luzin set $L \subseteq R$. So $L \cap N$ is countable for every nowhere-dense set N. Let $P = \{p \subseteq L \mid p \text{ is a finite free set for } f\}$. We will show now P satisfies the c.a.c. Let $\{p_\alpha \mid a < \omega_1\} \subseteq P$ be given. As before (Theorem 2.2) assume, w.l.o.g. that the p_α are pairwise disjoint and of the same cardinality n. Say $p_\alpha = \{a_1^\alpha, \cdots, a_n^\alpha\}$ enumerated in increasing order. As $f(r)$ is closed, we can find rational intervals $I_1, \cdots, I_n$ (not depending on α, without loss of generality) such that $a_i^\alpha \in I_i$ and for $j \neq i$, $I_i \cap f(a_j^\alpha) = \emptyset$. So again, if p_α and p_β are incompatible, it is only because for some $1 \leqq i \leqq n$, $\{a_i^\alpha, a_i^\beta\}$ is not free. Again we define by induction on $i \leqq n$ uncountable sets A_i, $B_i \subseteq \omega_1$ such that

$$\alpha \in A_i, \beta \in B_i \Rightarrow \{a_1^\alpha, \cdots, a_i^\alpha\} \cup \{a_1^\beta, \cdots, a_i^\beta\} \text{ is free.}$$

$A_0 = B_0 = \omega_1$. Assume A_{i-1}, B_{i-1} are defined. Given any uncountable set $A \subseteq R$ we can, by throwing countably many reals out of A, assume the following hold: For any (rational) interval I, if $A \cap I \neq \emptyset$ then $A \cap I$ is uncountable. So we stipulate all of our uncountable sets of reals have this property. For each a_i^β, $\beta \in B_{i-1}$ pick a rational interval I_β such that

$$(A = \{a_i^\alpha \mid \alpha \in A_{i-1}\}) \cap I_\beta \neq \emptyset \quad \text{and} \quad I_\beta \cap f(a_i^\beta) = \emptyset.$$

This is possible to find because $A \subseteq L$ is uncountable, hence of second category, and $f(r)$ is closed and nowhere dense. Now for some interval I, $I_\beta = I$ for uncountably many $\beta \in B_{i-1}$, say B'_{i-1} is the collection of all these indexes. $A \cap I \neq \emptyset$ hence even uncountable. Repeat now the other way around; for each $a_i^\alpha \in A \cap I$ find J_α a rational interval with $f(a_i^\alpha) \cap J_\alpha = \emptyset$ and $J_\alpha \cap \{a_i^\beta \mid \beta \in B'_{i-1}\} \neq \emptyset$ (hence uncountable). So we get an interval J and $A_i \subseteq A_{i-1}$ uncountable such that for $\alpha \in A_i$, $a_i^\alpha \in A \cap I$ and $J_\alpha = J$. Define $B_i \subseteq B'_{i-1}$, uncountable such that $\beta \in B_i \Rightarrow a_i^\beta \in J$. □

Now we iterate with finite support (i.e. taking direct limit at limit stages like [7]) forcing posets like P and the length of the iteration is ω_2, so that each intermediate stage is obtained by a c.a.c. poset of cardinality $\aleph_1$. Hence CH holds at each intermediate stage and Lemma 2.4 can be used. A problem might arise because in an ω_2 stage iteration one can do $\aleph_2$ many jobs while there are $2^{\aleph_2}$ many possible set mappings to deal with. This is why we have to use the diamond on $\aleph_2$ (on the set of ordinals of cofinality ω_1).

We give now more details. Let $\langle S_\alpha \mid \alpha < \omega_2 \rangle$ be in L a $\diamond$ sequence. So $S_\alpha \subseteq \alpha$ and for any $X \subseteq \omega_2$, $\{\alpha \mid \mathrm{cf}(\alpha) = \omega_1$ and $S_\alpha = X \cap \alpha\}$ is stationary in ω_2. $H(\aleph_2)$ is the collection of all sets which have transitive closure of cardinality $\leqq \aleph_1$. Let $F: \aleph_2 \to H(\aleph_2)$ be one to one and onto. We define the posets P_α, $\alpha < \omega_2$ such that $P_\alpha \in H(\aleph_2)$ (P_α is the α iteration). Suppose P_α is defined. Look at $F'' S_\alpha = \tau$. If τ is a name in L^{P_α} and $\emptyset \Vdash$ "τ is a nowhere-dense set mapping" then define $P_{\alpha+1} = P_\alpha * P$ where P is defined using Lemma 2.4 such that in $V^{P_{\alpha+1}}$ τ has an uncountable free set (if τ is not as above define $P_{\alpha+1}$ so that Martin's Axiom finally holds). Now let P_{ω_2} be the direct limit of P_α, $\alpha < \omega_2$. We have to show that in $V^{P_{\omega_2}}$ (*) holds. Well, let f be a nowhere-dense set mapping in $V^{P_{\omega_2}}$. We can assume that the values of f are *closed* nowhere-dense sets, and as a closed set can be described by a real we can assume f is a function from the reals into the reals. Now the reals of $V^{P_{\omega_2}}$ have names in $H(\aleph_2)$ (because P_{ω_2} satisfies the c.a.c.) so we can pick a function τ such that $\tau \subseteq H(\aleph_2)$ and such that for any name of a real, $r \in H(\aleph_2)$, $\tau(r)$ is another name of a real with $\emptyset \Vdash$ "$f(r) = \tau(r)$". If we let $\tau_\alpha = \tau \cap L^{P_\alpha}$ it follows that for a closed unbounded set C if $\alpha \in C$ and if $\mathrm{cf}(\alpha) = \omega_1$ then τ_α is in L^{P_α} a nowhere dense set mapping. $C' = \{\alpha \mid \tau \cap F''\alpha = \tau \cap L^{P_\alpha}$ or $\mathrm{cf}(\alpha) \neq \omega_1\}$ contains a closed unbounded subset of ω_2 (this follows from the following facts: $\tau \cap F''\alpha$ and $\tau \cap L^{P_\alpha}$ are of cardinality $\aleph_1$, they form an increasing sequence, continuous at ordinals of cofinality ω_1 and the union of these sequences is τ). Set $X = F^{-1}\tau$. Find $\alpha \in C \cap C'$ $\mathrm{cf}(\alpha) = \omega_1$ such that $S_\alpha = X \cap \alpha$, it follows that $F'' S_\alpha = F'' X \cap \alpha = \tau \cap F''\alpha = \tau \cap L^{P_\alpha} = \tau_\alpha$ is in L^{P_α} a name of a nowhere-dense set mapping, hence in $V^{P_{\alpha+1}}$ f has an uncountable free set. □

REMARKS. (a)† $2^{\aleph_0}$ can be given any desired value in Theorem 2.4. This by first adding κ many Cohen reals and then essentially repeat the iteration of 2.4, using $\diamond$ on κ. Observe that at limit states of cofinality ω_1, an ω_1 cofinal sequence of reals is a Luzin set.

(b) If Martin's Axiom holds then one can give a diagonalization argument, using the posets P_r, to construct a nowhere-dense set mapping which does not have any free set of cardinality continuum.

† Remarked by S. Todorčević and J. Steprāns.

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Department of Mathematics
Dartmouth College
Hanover, New Hampshire, USA